

Cálculo Integral. Aplicações

$$X \subset \mathbb{R}^n, \quad f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\int_X f \quad \equiv \quad \text{integral de } f \text{ em } X.$$

$$\underline{\mathbb{R}^2}: \quad \int_X f = \int_X \left(\int f(x,y) dx \right) dy$$

Integrais
múltiplos
iterados

$$= \int_X \left(\int f(x,y) dy \right) dx$$

$$\mathbb{R}^3: \quad \int_X f = \int_X \left(\int \left(\int f(x,y,z) dx \right) dy \right) dz$$

6 maneiras!

Caso importante: $f \equiv 1$

$$\int_X f = \int_X 1 = \text{vol}_n(X)$$

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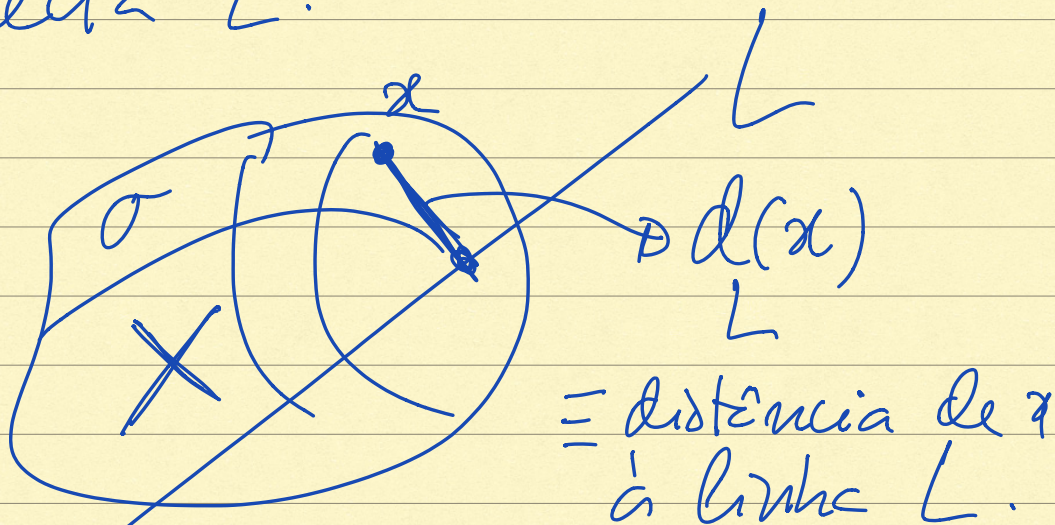
1) $f(x) = \sigma(x) \geq 0, x \in X$
Densidade de massa

$$\Rightarrow \int_X \sigma \equiv M \text{ (Massa)}$$

2) Centro de massa: (centroide)
 $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ $\sigma \equiv 1$

$$\bar{x}_j = \frac{1}{M} \int_X x_j \sigma$$

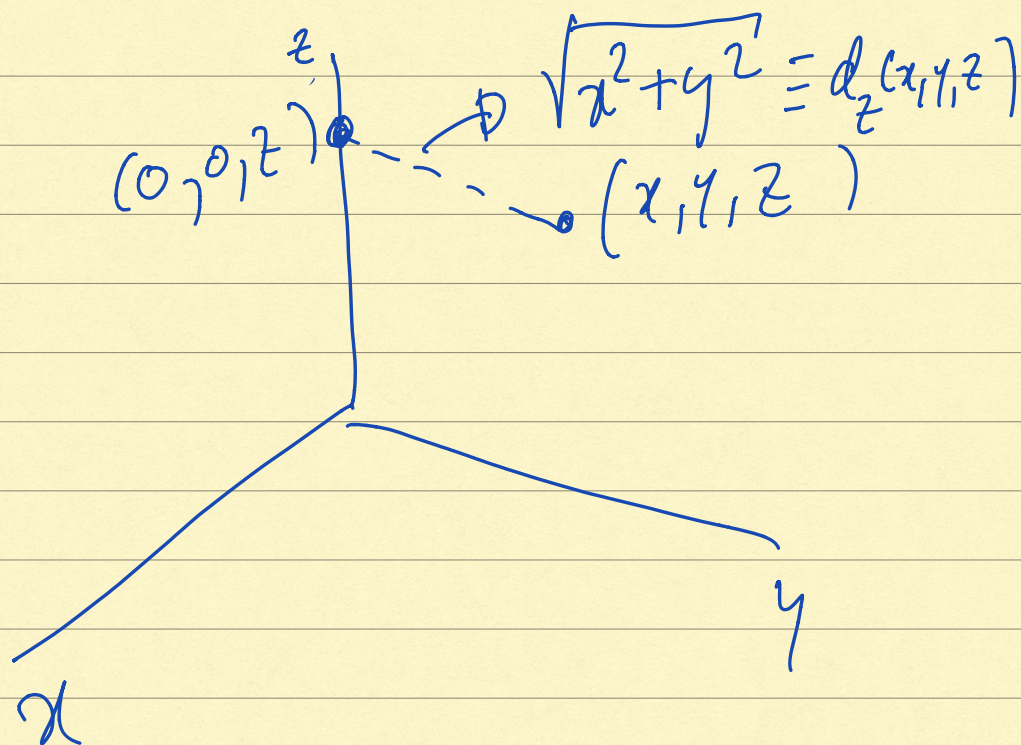
3) Momento de inércia de X relativo a uma linha recta L .



$$I_L(X) = \int_X \sigma d_L^2$$

Casos importantes: $L \equiv$ eixo o
 condensado

Exemplo: eixo oz $d_{oz}(x, y, z) = \sqrt{x^2 + y^2}$



Example: $X \subset \mathbb{R}^3$:

$$0 \leq x \leq 1; \quad \frac{x}{2} \leq y \leq x; \quad 0 \leq z \leq x$$

$$\sigma(x, y, z) = 1$$

$$I_z(X) = \iiint_X (x^2 + y^2) dx dy dz$$

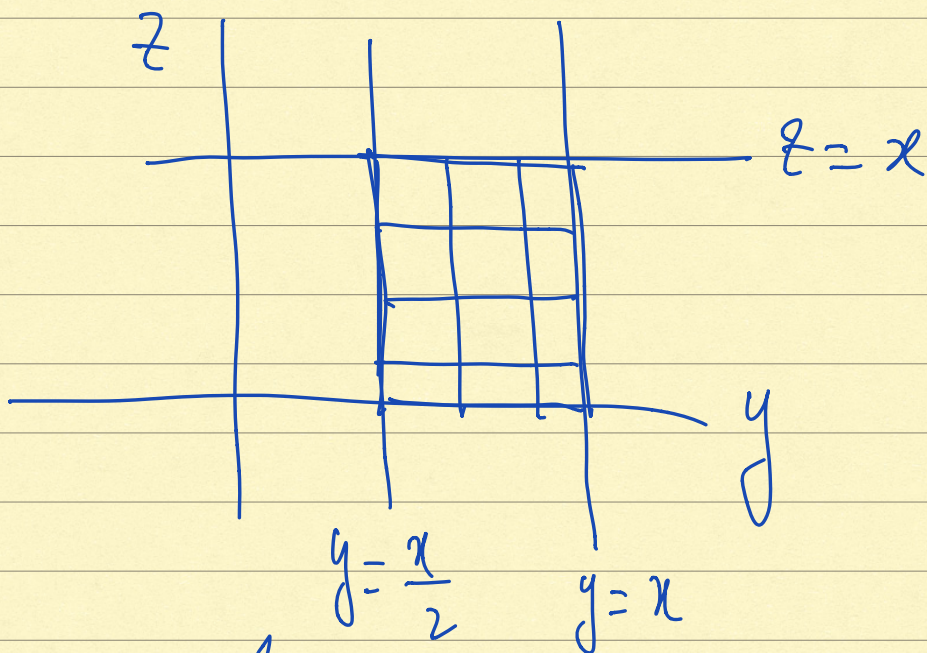
$$\underbrace{0 \leq x \leq 1}; \quad \frac{x}{2} \leq y \leq x; \quad \underbrace{0 \leq z \leq x}$$

$$dy \, dz \, dx$$

$$x \text{ fixo} \rightarrow z(x) \rightarrow y(x, z)$$

$$\boxed{dz \, dy \, dx}$$

$$0 \leq x \leq 1 \text{ fixo}$$



$$I_z(x) = \int_0^1 \left(\int_{\frac{x}{2}}^x \left(\int_0^x (x^2 + y^2) dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_{\frac{x}{2}}^x \left(x(x^2 + y^2) \right) dy \right) dx$$

etc...

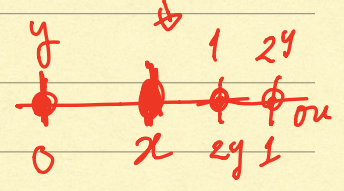
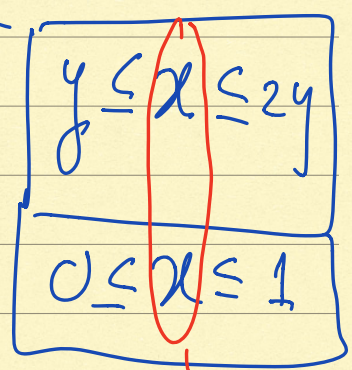
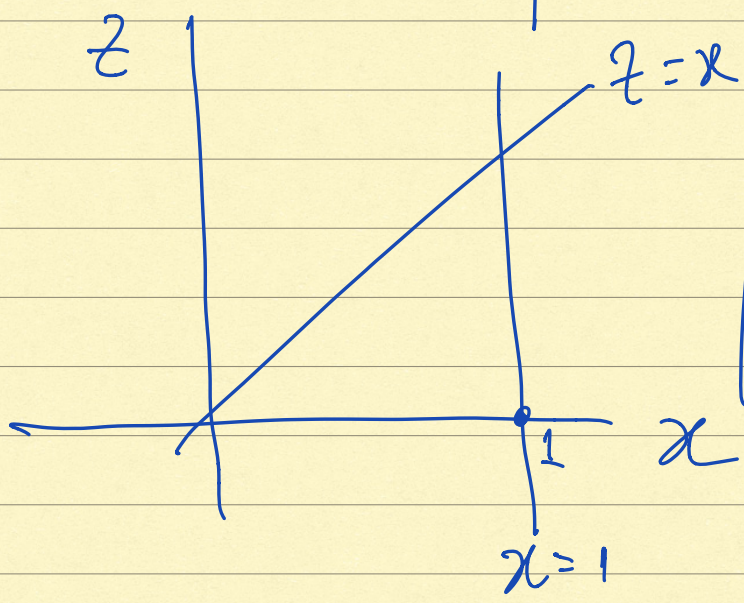
————— u —————

$dx dz dy$ $y \text{ fixed} \rightarrow z(y) \rightarrow x(y, z)$

$0 \leq x \leq 1$; $\frac{x}{2} \leq y \leq x$; $0 \leq z \leq x$

$0 \leq \frac{x}{2} \leq y \leq x \leq 1$

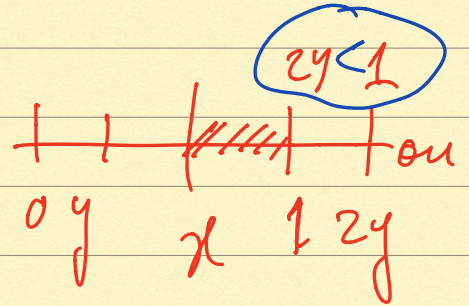
$0 \leq y \leq 1$



$0 \leq y \leq 1$
fixo

$$y \leq x \leq 2y$$

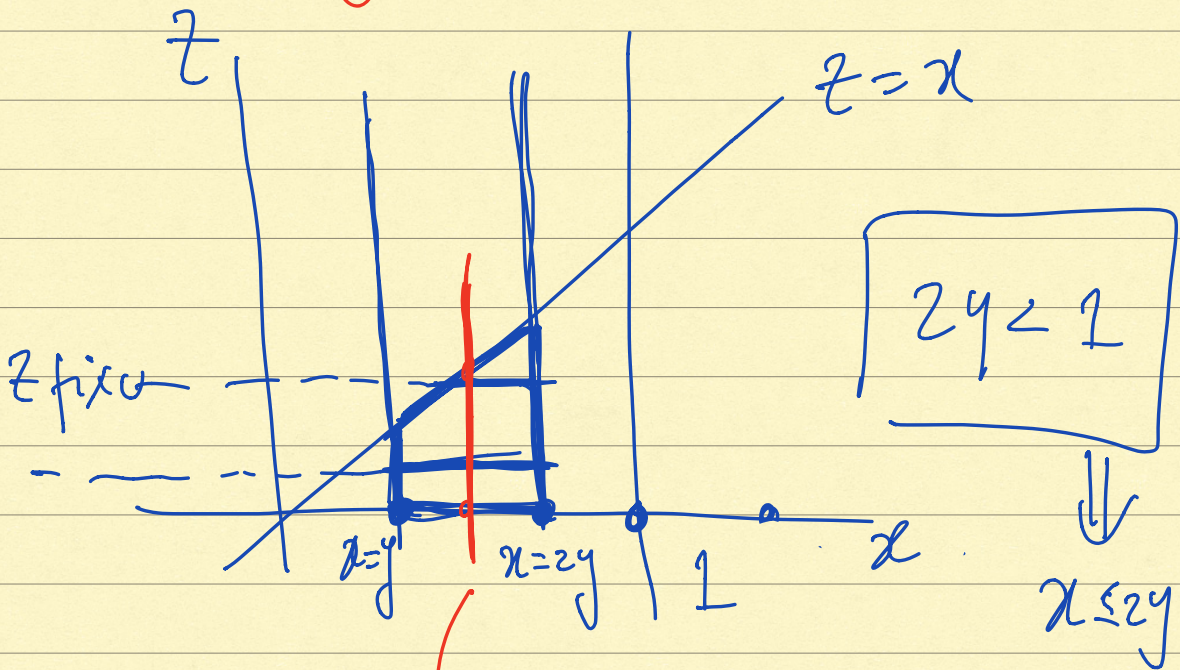
$$0 \leq x \leq 1$$



$$y \leq x \leq 2y$$

$$x \leq 1$$

2 casos

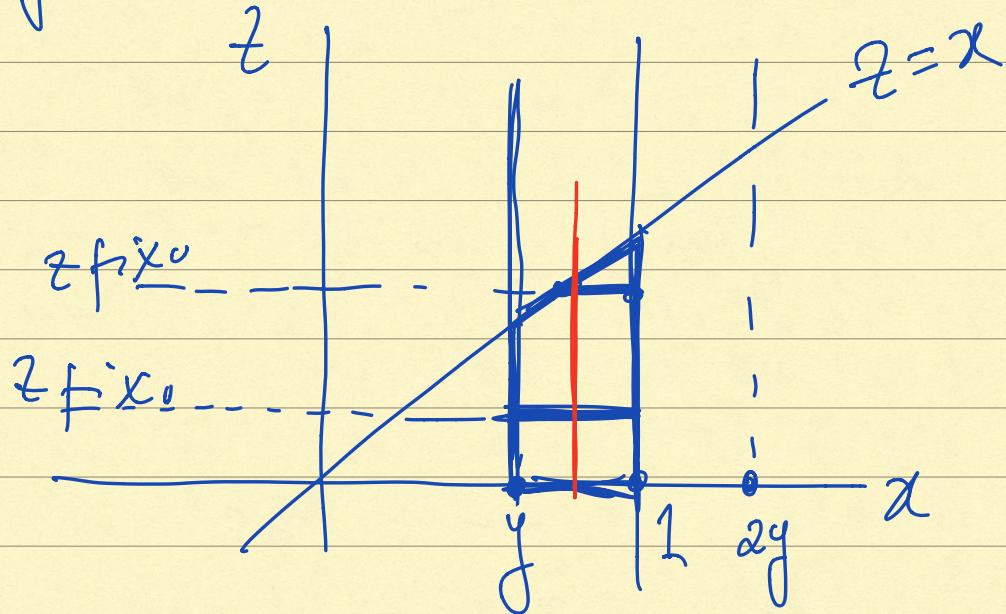


$\int dx dz dy$
 $y < \frac{1}{2}$

→ 2 integrales

$dz dx dy$

$$2y > 1 :$$



$$\int_{y > \frac{1}{2}} dz dx dy \Rightarrow 2 \text{ integrals}$$

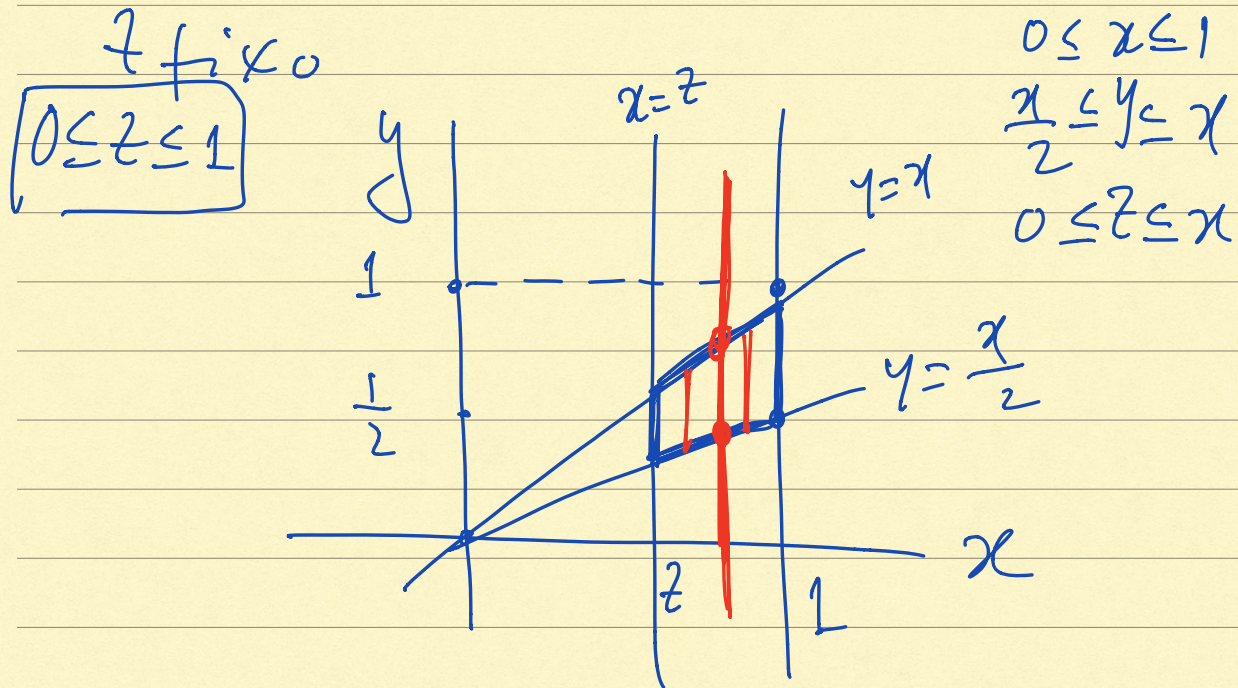
$$\therefore \int dz dx dy \rightarrow 4 \text{ integrals triples}$$

$$\int dz dx dy \rightarrow 2 \text{ integrals triples}$$

Ord y dz

or

dy dz



$dy dz dz$

$$\text{Vol}_2(x) = \int_0^1 \left(\int_z^1 \left(\int_{\frac{x}{2}}^x dy \right) dx \right) dz$$

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$dz dy dz \rightarrow 6$ integrals!
(Exercício)

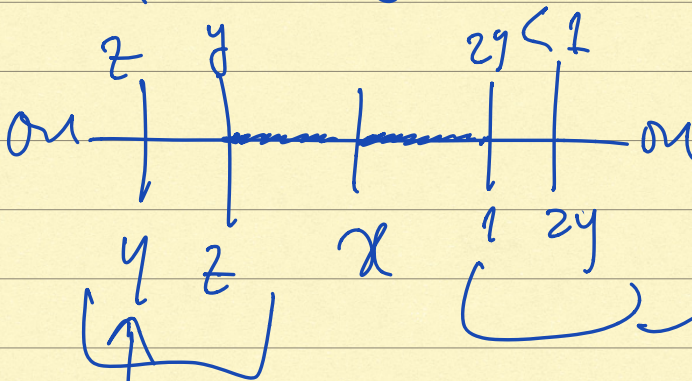
Ajuda para $dx dy dz$:

$$0 \leq x \leq 1; \quad \frac{x}{2} \leq y \leq x; \quad 0 \leq z \leq x$$

z fixo

$$0 \leq z \leq 1$$

$$\begin{cases} y \leq x \leq 2y \\ z \leq x \leq 1 \end{cases}$$



$$y \leftrightarrow \frac{1}{2}$$

$$z \leftrightarrow \frac{1}{2}$$

etc...